

CAUSAL MONOTONICITY, OMNISCIENT FOLIATIONS, AND THE SHAPE OF SPACE

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ABSTRACT. What is the shape of space in a spacetime? One way of addressing this issue is to consider edgeless spacelike submanifolds of the spacetime. An alternative is to foliate the spacetime by timelike curves and consider the quotient obtained by identifying points on the same timelike curve. In this article we investigate each of these notions and obtain conditions such that it yields a meaningful shape of space. We also consider the relationship between these two notions and find conditions for the quotient space to be diffeomorphic to any edgeless spacelike hypersurface. In particular, we find conditions in which merely local behavior (being spacelike) combined with the correct behavior on the homotopy level guarantees that a putative shape of space really is precisely that.

0. INTRODUCTION

What is the shape of space?

There are two questions comprised in that query: First, what is, or ought to be, meant by the phrase, “shape of space”? Second, assuming we know what this means, what shape does space have?

By “shape of space”, we will mean (roughly) the diffeomorphism class of any (sufficiently well-behaved) edgeless spacelike hypersurfaces in a spacetime, assuming there is only one such class; this clearly begs the question of which spacetimes are so favored as to have a shape of space in this sense. Part of the purpose of this paper is to give a good indication of a large class of spacetimes having a well-defined shape of space, namely, those with a distinguished class of timelike curves foliating the spacetime, obeying some mild restrictions. Given such a spacetime, we can address ourselves to the second question: The shape of space is found to be the leaf space of the foliation.

It should be noted that everything in this paper is conformally invariant: This is an inquiry into the global causal structure of spacetimes.

It needs to be emphasised that the shape of space considered here is generally quite different from the concept of a Cauchy surface. In part this is because the spacetimes considered in this paper are not restricted to being globally hyperbolic, which is a prerequisite for having a Cauchy surface. Furthermore, a Cauchy surface need not be a shape of space: the de Sitter spacetime has Cauchy surfaces which

1991 *Mathematics Subject Classification.* 53C, 83C.

Key words and phrases. spacetime, Hausdorff, global structure, timelike foliation, orbit space, leaf space, shape of space.

Support from the Aspen center for Physics is gratefully acknowledged.

are spheres but also has perfectly well-behaved edgeless spacelike hypersurfaces diffeomorphic to \mathbb{R}^3 . A rough and ready rule: A spacetime with a timelike boundary cannot have a Cauchy surface (but may have a shape of space); a spacetime with a spacelike boundary cannot have a shape of space (but may have a Cauchy surface). It is only in a spacetime with only null boundaries that the two might coexist (and then must be homeomorphic).

Section 1 explores some general results for foliations of spacetimes by timelike curves; it turns out that the crucial matter is whether or not the leaf space is Hausdorff. Section 2 explores how this is related to the concept of causal monotonicity for a complete timelike vector field tangent to the foliation. Section 3 makes use of the notion of an omniscient foliation and specializes to the case of conformastationary spacetimes. Section 4 treats an application to strongly causal spacetimes, giving conditions under which all locally “nice” spacelike hypersurfaces are globally well-behaved (and so have the proper shape of space).

Several of the results in this paper were announced, in preliminary form, in [HL].

1. TIMELIKE FOLIATIONS

Throughout this paper, V will denote a spacetime (a time-oriented Lorentz manifold: paracompact and Hausdorff), \mathcal{F} a differentiable foliation of V by timelike curves, and Q the leaf space of \mathcal{F} , i.e., the quotient space (with quotient topology) of V modulo the equivalence relation that two points are related iff they lie on the same leaf (i.e., curve) in \mathcal{F} ; in short, $Q = V/\mathcal{F}$.

A typical example of a timelike foliation on a spacetime V is the set of integral curves of some timelike vector field U . In particular, if U is complete (i.e., each integral curve γ is defined as a map from all of \mathbb{R} into V), then there is an induced group action of \mathbb{R} on V : For any $x \in V$, let γ_x be the integral curve of U with $\gamma_x(0) = x$; then for $t \in \mathbb{R}$, $t \cdot x = \gamma_x(t)$. This is a convenient way to have things arranged because then \mathcal{F} consists of the orbits of the group action, and $Q = V/\mathbb{R}$, the orbit space. In fact, so long as V is chronological (no closed timelike loops), any timelike foliation can be expressed in this manner:

Lemma 1.1. *If \mathcal{F} is a timelike foliation in a chronological spacetime V , then there is a complete vector field whose integral curves are \mathcal{F} .*

Proof. First observe that since V is chronological, the leaves of \mathcal{F} are all diffeomorphic to the line, not the circle. Next, observe that any such foliate γ must exit every compact set K : Otherwise, there must be some point p in K which is not on γ but is an accumulation point of γ . Consider a “flow box” around p : a (small) embedded hypersurface P through p , transverse to \mathcal{F} , and for each x in P a piece γ_x of the foliate through x , such that $\bigcup\{\gamma_x \mid x \in P\}$ is a neighborhood of p . Note that for all x in P sufficiently close to p , γ_x must intersect both the future and the past of p (γ_p surely does so, and γ_x is close to γ_p for x close to p). We have that for some sequence of points $\{x_n\}$ in P approaching p , $\gamma^n = \gamma_{x_n}$ is actually a part of γ (so that p can be approached by γ). Restricting attention to n being sufficiently large, we can find q_n and r_n as points on γ^n respectively to the future and past of p . We have for all such n and m , all of γ^n is to the future of all of γ^m or *vice versa*, depending on how the segments are situated on γ . Assuming the former, then we have $p \gg r_n \gg q_m \gg p$, violating the chronology condition.

Since V is time-oriented, \mathcal{F} is orientable; therefore, we can pick a differentiable

vector field U everywhere tangent to \mathcal{F} . We will construct a positive scalar function $\lambda : V \rightarrow \mathbb{R}$ such that λU is complete.

Each foliate has a parametrization making it an integral curve of U ; there is ambiguity in this parametrization (up to an additive constant), but we don't care whether or not it can be done in a globally continuous manner. Note that λU will be complete iff for every foliate $\gamma : (a, b) \rightarrow \mathbb{R}$, $\int_t^b (1/\lambda) \circ \gamma = \infty$ and $\int_a^t (1/\lambda) \circ \gamma = \infty$, where t is any number in the domain. Now let $\{K_n\}$ be an exhaustion of V by compact sets: K_n is a subset of the interior of K_{n+1} and $\bigcup_{n=1}^{\infty} K_n = V$ (since V is chronological, it is not compact, so this can be done with an infinite family of compact sets). For each n , there is some $\delta_n > 0$ such that for every foliate γ intersecting K_n , the parameter for γ increases by at least δ_n as γ extends from where it exits K_n for the last time to where it exits K_{n+1} for the first time. Now let λ be any positive scalar function on V such that on $K_{n+1} - K_n$, $\lambda < \delta_n$. Then over each of the intervals mentioned above, $\int (1/\lambda) \circ \gamma > 1$; therefore, $\int_t^b (1/\lambda) \circ \gamma = \infty$. Similarly, we can arrange it so that $\int_a^t (1/\lambda) \circ \gamma = \infty$. \square

From now on, we will take \mathcal{F} to be given as the orbit space of an \mathbb{R} -action, with the corresponding vector field U being future-directed. Note that this makes V the total space of a principal line bundle over Q . Now, any line bundle (with a real action) over a true manifold (Hausdorff and paracompact) admits a global cross-section (see [KN], Theorem I.5.7), so the total space is diffeomorphic to $\mathbb{R} \times (\text{base space})$ (see, e.g., [H1, Corollary 1]). In [H1, Theorem 2] it was shown Q must be a *near manifold*: a topological space with a differentiable atlas and a countable basis; only Hausdorffness is lacking in order for it to be a true manifold (paracompactness would follow from Hausdorffness).

Here's a typical example with a non-Hausdorff leaf space: With \mathbb{L}^n denoting Minkowski n -space, with orthogonal coordinates $\{t, x^1, \dots, x^{n-1}\}$, let $V = \mathbb{L}^3 - \{(0, 0, 0)\}$. For each $(x, y) \neq (0, 0) = \mathbf{0}$, let $L_{x,y}$ be the line $\{x^1 = x, x^2 = y\}$; let $L_{\mathbf{0}}^+$ and $L_{\mathbf{0}}^-$, respectively, denote the $t > 0$ and $t < 0$ half-lines making up $\{x^1 = 0, x^2 = 0\}$. Then let $\mathcal{F} = \{L_{x,y} \mid (x, y) \neq (0, 0)\} \cup \{L_{\mathbf{0}}^+, L_{\mathbf{0}}^-\}$. The points of the leaf space Q can clearly be labeled by $\{(x, y) \mid (x, y) \neq \mathbf{0}\} \cup \{\mathbf{0}^+, \mathbf{0}^-\}$. The neighborhoods of each (x, y) are given by the "tubular" (i.e., \mathbb{R} -invariant) neighborhoods of $L_{x,y}$; these are just reflections of the neighborhoods of (x, y) in the plane, \mathbb{R}^2 . However, the tubular neighborhoods of $L_{\mathbf{0}}^+$ and of $L_{\mathbf{0}}^-$ all intersect one another (alternatively stated: any set of foliates forming a neighborhood of a point on $L_{\mathbf{0}}^+$ must overlap with any set of foliates doing similar duty for $L_{\mathbf{0}}^-$); it follows that, in the quotient topology on Q , all neighborhoods of $\mathbf{0}^+$ and of $\mathbf{0}^-$ intersect with one another: $\{\mathbf{0}^+, \mathbf{0}^-\}$ form a *non-Hausdorff pair*. Thus, Q is a classic near-manifold: \mathbb{R}^2 with the origin replaced by a double-point. Note that, in particular, $V \neq \mathbb{R} \times Q$, since V is Hausdorff, while Q is not.

There is a particular feature of the causal structure of \mathcal{F} in the example above that is worth calling attention to: The foliates corresponding to the non-Hausdorff pair in Q , i.e., $L_{\mathbf{0}}^+$ and $L_{\mathbf{0}}^-$, have the property that any point in the first is to the future of any point in the second; in general, when this happens, we will say that the second foliate is *ancestral* to the first, and we will call the two foliates an *ancestral pair*. In fact, this is the key to discovering that a leaf space is Hausdorff (announced as Theorem 1 of [HL]):

Theorem 1.2. *Let V be a chronological spacetime and let \mathcal{F} be a foliation of V by*

timelike curves. If \mathcal{F} contains no ancestral pairs, then the leaf space Q is Hausdorff; hence, Q is a true manifold and $V \cong \mathbb{R} \times Q$ (as a manifold).

Proof. Suppose Q contains a non-Hausdorff pair, $\{q, q'\}$; let $\{\gamma, \gamma'\}$ be the corresponding foliates in \mathcal{F} . To say that all neighborhoods in Q of q and q' intersect is to say that all tubular neighborhoods in V of γ and γ' have a foliate in common. Pick points p and p' respectively on γ and γ' . By considering fundamental neighborhood systems of p and p' and extending these (by the \mathbb{R} -action) to tubular (i.e., \mathbb{R} -invariant) neighborhoods, we deduce the existence of a family of foliates $\{\gamma_n\}$ and numbers $\{t_n\}$ and $\{t'_n\}$ such that $\{\gamma_n(t_n)\}$ approaches p and $\{\gamma_n(t'_n)\}$ approaches p' .

As in the proof of Lemma 1.1, we form flow boxes W and W' around, respectively, p and p' ; since V is Hausdorff, we can do this so that W and W' do not intersect. As in that proof, we observe that for n large enough, γ_n enters both the past and future of p within W , and of p' within W' ; say $\gamma_n(t_n + \delta_n) \gg p \gg \gamma_n(t_n - \delta_n)$ and $\gamma_n(t'_n + \delta_n) \gg p' \gg \gamma_n(t'_n - \delta_n)$. Now either, for infinitely many n , $t_n > t'_n$, or, for infinitely many n , $t_n < t'_n$ (or possibly both). Assume the former is true; then, since the segments of γ_n in W and W' don't overlap, we also have (for infinitely many n) $t_n - \delta_n > t'_n + \delta_n$. Thus (for those n), $p \gg \gamma_n(t_n - \delta_n) \gg \gamma_n(t'_n + \delta_n) \gg p'$.

It follows that for all t and t' , $\gamma(t)$ and $\gamma'(t')$ are timelike-related. Let $A = \{(t, t') \mid \gamma(t) \gg \gamma'(t')\}$ and $B = \{(t, t') \mid \gamma(t) \ll \gamma'(t')\}$. Note that A and B are both open subsets of \mathbb{R}^2 ; since they are disjoint and their union is the plane, it follows that one of them is empty, the other all of \mathbb{R}^2 . That says precisely that one of the two foliates is ancestral to the other. \square

It should be noted that an ancestral pair of leaves is not incompatible with a Hausdorff leaf space: Let $V = \{(t, x) \in \mathbb{L}^2 \mid 2x < t < 2x + 1\}$, with \mathcal{F} given by the integral curves of $\partial/\partial t$. Note, for instance, that $\{(t, 0) \mid 0 < t < 1\}$ is ancestral to $\{(t, 1) \mid 2 < t < 3\}$; yet the leaf space is just \mathbb{R}^1 . Whenever this situation arises—that \mathcal{F} contains an ancestral pair, but Q is Hausdorff—any lift of Q in $V \cong \mathbb{R} \times Q$ is of necessity not achronal.

In spite of its being a condition that is stronger than necessary, the absence of ancestral pairs is a very convenient criterion to use. For instance, here is a property of a foliation \mathcal{F} (to be used in the next section) that implies no ancestral pairs in \mathcal{F} (so that Theorem 1.2 can be invoked): that each foliate in \mathcal{F} enter the future of each point in V . If we think of the foliate as an observer, then this says the observer (eventually) sees every event in the spacetime. Thus, let us call a timelike curve γ in a spacetime V *past-omniscient* if $I^-(\gamma) = V$ (and *future-omniscient* if $I^+(\gamma) = V$); in the usage of [HE], these are the same, respectively, as having no future and no past event horizon. We will call a timelike foliation past-omniscient (respectively, future-omniscient) if all the foliates are past-omniscient (respectively, future-omniscient); we call it *half-omniscient* if it is either past- or future-omniscient. (Full omniscience—being both past- and future-omniscient—will be used in Section 3.)

A simple example of a half-omniscient spacetime: Consider the warped product spacetime $(\alpha, \omega) \times K$ for $-\infty \leq \alpha < \omega \leq \infty$ and (K, h) any Riemannian manifold, with metric $-(dt)^2 + r(t)^2 h$ for some positive function r . (For instance, Robertson-Walker spaces are of this form, with (K, h) of constant curvature.) Let $U = \partial/\partial t$; then U is future-complete (i.e., its orbits extend to ∞ in the future, taken to be the positive- t direction) if and only if $\int_c^\omega 1/r = \infty$ for some c between α and ω .

With \mathcal{F} the foliation given by the orbits of U , \mathcal{F} is past-omniscient if and only if U is future-complete. (This is most easily seen in the conformally equivalent metric $-(dt/r(t))^2 + h$, which yields a simple static spacetime; omniscience is conformally invariant, so this metric provides an easy venue to check on it.)

Proposition 1.3. *Let V be a chronological spacetime and let \mathcal{F} be a foliation of V by timelike curves. If \mathcal{F} is half-omniscient, then it has no ancestral pairs.*

Proof. Assume \mathcal{F} is past-omniscient. Let γ and γ' be any two foliates. If γ is ancestral to γ' , then consider any point p' on γ' . Being past-omniscient, γ contains some point $p \gg p'$; however, by ancestry, $p \ll p'$, violating chronology. Thus, \mathcal{F} has no ancestral pairs. \square

Although, as demonstrated by the example following Theorem 1.2, non-ancestry of foliates is not equivalent to Hausdorffness of the leaf space, it is interesting to note that non-ancestry is equivalent to Hausdorffness in a larger curve space, one which contains the leaf space: the space $\mathcal{C}(V)$ of smooth endless causal paths (i.e., unparametrized curves) of V equipped with an appropriate topology. If V is a strongly causal spacetime, then the compact-open topology is appropriate, but for a space-time satisfying only the chronological condition, more subtlety is required.

The appropriate topology to use for $\mathcal{C}(V)$ may be called the *interval topology*: For any collection of open sets $\{W_1, \dots, W_n\}$ in V , let $[W_1, \dots, W_n] = \{c \in \mathcal{C}(V) \mid \text{for some interval } J \text{ of } c, J \subset \bigcup W_i \text{ and, for all } i, J \text{ intersects } W_i\}$ (by interval is meant a connected subset of a path); then the collection of all possible $[W_1, \dots, W_n]$ is a sub-basis for the interval topology. Note that this is, in general, finer than the compact-open topology (which has as a sub-basis all such sets as these with $n = 1$). However, if V is strongly causal, then the two topologies coincide. This is demonstrated by showing that in a strongly causal spacetime, convergence of paths in the compact-open sense implies convergence through connected intervals, as can be seen by refining a chain of open sets to a chain of simply interlocking causally convex sets:

For any $c \in [W_1, \dots, W_n]$, we can cover J (the relevant interval of c) with a finite chain of causally convex open sets U_1, \dots, U_m such that each U_i intersects J in some $c(s_i)$ with $s_i < s_{i+1}$, each $U_i \cap U_{i+1} \neq \emptyset$, each $U_i \cap U_j = \emptyset$ if $|i - j| > 1$, $\bigcup U_i \subset \bigcup W_j$, and each W_j contains some $U_i \cap U_{i+1}$. For each $i < m$, $U_i \cap U_{i+1}$ contains a point of c (since c must enter U_{i+1} , and only U_i can contain that point of entry); thus, $c \in \bigcap [U_i \cap U_{i+1}]$, an open set in the compact-open topology. Furthermore, consider any $c' \in \bigcap [U_i \cap U_{i+1}]$: With $c'(t_i)$ the point in $U_i \cap U_{i+1}$, on the interval $[t_i, t_{i+1}]$, c' remains within U_{i+1} ; thus, on $[t_1, t_{m-1}]$, c' remains within $\bigcup U_i$. Therefore, $\bigcap [U_i \cap U_{i+1}] \subset [W_1, \dots, W_m]$, which is a subset of $[W_1, \dots, W_n]$.

Thus, the following proposition is an extension of [L, Proposition 4.3], which treats the strongly causal case in the compact-open topology. Theorem 1.2 tells us that non-ancestry of a pair of foliates implies they are Hausdorff separated in Q . This proposition embeds Q in $\mathcal{C}(V)$ (so that two elements of Q which are Hausdorff separated in $\mathcal{C}(V)$ are *a fortiori* Hausdorff separated in Q); and it also tells us that non-ancestry of a pair of foliates is equivalent to their being Hausdorff separated in that larger space.

Proposition 1.4. *For any chronological space-time V , $\mathcal{C}(V)$ is Hausdorff (in the interval topology) iff it contains no ancestral pairs, in that two curves comprise an ancestral pair if and only if they are not Hausdorff separated. Irrespective of*

ancestry, for any foliation \mathcal{F} of V by timelike curves, the leaf space Q is topologically embedded in $\mathcal{C}(V)$ (with the interval topology).

Proof. Suppose that c is ancestral to c' . Let both curves be parametrized by \mathbb{R} and future-directed. For all n , let c_n consist of the concatenation of $c|_{(-\infty, n]}$, a future-timelike curve from $c(n)$ to $c'(-n)$, and $c'|_{[-n, \infty)}$, smoothed out at the corners. Then in the interval topology, both c and c' are limits of the sequence $\{c_n\}$; hence, $\mathcal{C}(V)$ is not Hausdorff.

Conversely, let $\{c, c'\}$ be a non-Hausdorff pair in $\mathcal{C}(V)$. Let p and p' be any points on c and c' respectively, and let $\{W_k\}$ and $\{W'_k\}$ be respective (disjoint) fundamental neighborhood systems. Let W_k^+ denote $W_k \cap I^+(p)$, and similarly for $-$ and primed. For each k there is a $c_k \in [W_k^-, W_k, W_k^+] \cap [W'_k^-, W'_k, W'_k^+]$. Each c_k passes from the past to the future of p before (or after) encountering W'_k , and similarly for p' and W_k . Therefore, the same procedure as used in the proof of Theorem 1.2 shows that c and c' form an ancestral pair.

Let $\pi : V \rightarrow Q$ be the natural projection, and let $j : Q \rightarrow \mathcal{C}(V)$ be the mapping sending $\pi(p)$ to the foliate p lies on. The quotient topology on Q consists of all $\pi(W)$ for W a tubular open set in V . If W is any tubular open set in V , then $j(\pi(W))$ is precisely $[W] \cap j(Q)$. Conversely: For any collection of open sets W_1, \dots, W_n in V , for any foliate $\gamma \in [W_1, \dots, W_n]$, there is a tubular neighborhood W of γ such that, in a neighborhood of the relevant interval J of γ , W lies inside $\bigcup W_i$. Then $j(\pi(W))$ is a quotient-neighborhood of γ lying inside $[W_1, \dots, W_n]$. \square

Thus a foliation \mathcal{F} is Hausdorff as a subspace of $\mathcal{C}(V)$ if and only if it has no ancestral pairs. But this really does require the interval topology, and not the compact-open topology, on $\mathcal{C}(V)$ if V is not strongly causal, as shown by this example:

Let V be the acausal flat cylinder: \mathbb{L}^2/\mathbb{Z} , where \mathbb{Z} acts on \mathbb{L}^2 via $n \cdot (t, x) = (t+n, x+n)$; take the foliation \mathcal{F} to be given by the integral curves of the vector field $\partial/\partial t$ (which is invariant under the action). Although this spacetime has closed null curves, it is chronological. Another way of viewing the spacetime is as the region $\{(t, x) \mid 0 \leq x \leq 1\}$ of \mathbb{L}^2 with $(t, 0)$ identified with $(t+1, 1)$; then the foliates are lines $\{x = x_0\}$ for $0 \leq x_0 < 1$. This easily shows that there are no ancestral pairs, and that the leaf space Q is a circle, \mathbb{S}^1 . However, \mathcal{F} is not Hausdorff as a subspace of $\mathcal{C}(V)$ in the compact-open topology (as modified for a space of unparametrized paths):

Let γ be the $\{x = 0\}$ foliate and let γ' be the $\{x = .5\}$ foliate. In the (modified) compact-open topology, a typical neighborhood of γ would be the set of all causal curves which pass through a given open set that γ passes through—say, for the point $p = [(0, 0)]$ (where $[\cdot]$ denotes equivalence class), all causal curves passing through some neighborhood U of p . Consider a similar neighborhood U' of a point $p' = [(.5, t_0)]$ on γ' ; we must show that there is a causal curve c which intersects both U and U' . Let q'_1 be the point on γ' just to the null-future of p , and q'_2 the point on γ' just to the null-past of p , i.e., $q'_1 = [(.5, .5)]$ and $q'_2 = [(.5, -.5)]$. If p' lies to the future of q'_1 (i.e., $t_0 > .5$), then $p \prec q'_1 \ll p'$, so $p \ll p'$, so there is a future-timelike curve c from p to p' ; and c clearly intersects both U and U' . Similarly if p' occurs on γ' to the past of q'_2 : $p \succ q'_2 \gg p'$, and there is a past-timelike curve from p to p' . If $q'_2 \prec p' \prec q'_1$ ($|t_0| \leq .5$), then let c be the curve $\{t = 2(1-t_0)(x-1)+1\}$ for $.5 \leq x \leq 1$; this is a future-causal curve from p' to p . Thus, in all cases, an element of $\mathcal{C}(V)$ lies in both neighborhoods.

2. CAUSAL MONOTONICITY

We now consider the nature of the timelike vector field U which generates the foliation \mathcal{F} . We will introduce a condition on U that ensures that no two foliates are ancestrally related, and that therefore the spacetime V is topologically $\mathbb{R} \times Q$; this will extend the result of Corollary 1 in [H1], which says that if a chronological spacetime with spacetime metric g has a complete timelike vector field U satisfying $\mathcal{L}_U g = \lambda g$ for λ non-negative (or non-positive), then it has that same topological form.

In the sequel, we will use $\{R_t\}$ to denote the flow of the vector field U , and $t \cdot p = R_t(p)$ (as indicated in section 1) for the point obtained by travelling a parameter distance t along the integral curve through a point p in V .

We want to generalize the behavior of a timelike conformal-Killing vector field, in its effect on causal curves; it turns out that this can be modeled fairly well by the infinitesimal effect of the field on the metric. Note that a conformal-Killing field has a conformal flow, which preserves causal character; for infinitesimal effects, note that the Lie derivative of the metric along such a field is proportional to the metric, so vanishes on any null vector. The generalizations of these properties in either of two directions (past or future) will be called “causal monotonicity”; for generalization in just one direction we will use the terms “causally decreasing” or “causally increasing”:

Definition. A diffeomorphism $f : V \rightarrow V$ is *causally decreasing* if it maps any causal curve to a causal curve and any timelike curve to a timelike one (preserving past/future), and *strictly causally decreasing* if it maps any causal curve to a timelike curve (preserving past/future).

It should be clear that the properties above are equivalent to the differential f_* mapping causal vectors to causal, and so on. Furthermore, this is equivalent to f_* just carrying null vectors to, respectively, causal or timelike vectors: A future-timelike vector X can be characterized as a vector which is expressible as a linear combination $aN+bL$ where N and L are future-null vectors and a and b are positive; thus, f_*N and f_*L future-causal implies f_*X future-timelike.

Definition. Let g be the spacetime metric, with the convention $g(X, X) \leq 0$ for causal vectors X ; a vector field U on V is *causally decreasing* if for any null vector N , $(\mathcal{L}_U g)(N, N) \leq 0$, and *strictly causally decreasing* if for any null vector N , $(\mathcal{L}_U g)(N, N) < 0$.

It was shown in [H2] that behavior of a timelike vector field in the non-strict sense above is equivalent to the same in that of its flow; in the strict sense, the behavior of the flow yields the behavior of the vector field (in that article, “monotonic” was used where here “decreasing” is used). For completeness, we restate this result and sketch the proof here:

Theorem 2.1. *A timelike vector field U is causally decreasing iff its flow $\{R_t\}$ in the forward direction (i.e., $t > 0$) is causally decreasing; if the vector field is strictly causally decreasing, then so is its forward flow.*

Proof. Its easy to see that if the flow of U is causally decreasing, then so is U : For any null vector N at p , just consider the extension of N to a flow-invariant field along the integral curve γ_p ; then $(\mathcal{L}_U g)_p(N, N) = (d/dt)g(N_t, N_t) \mid_{t=0}$ (where

$N_t = N_{\gamma_p(t)}$). If the flow is causally decreasing, then N remains causal, so that derivative is nonpositive, giving the result.

For the converse, let N be an arbitrary null vector at a point p , and again extend this to a flow-invariant field along γ_p . First, let us assume the stronger hypothesis, that U is strictly causally decreasing. As before, $(\mathcal{L}_U g)_t(N, N) = (d/ds)(g(N_s, N_s))|_{s=t}$; thus, U strictly causally decreasing at 0 (where N is null) implies N_t is timelike for small $t > 0$ and spacelike for small $t < 0$. In fact, N_t must be timelike for all $t > 0$, since otherwise there is some first point $t_0 > 0$ at which it is null, and the same argument then applies to N_{t_0} —but N_t cannot be spacelike for t just less than t_0 . (This establishes the last clause in the theorem.)

Next, assume only that U is causally decreasing. Then we can approximate U by a sequence of strictly causally decreasing vector fields, $\{U_n = (1 + (1/n)\lambda)U\}$ for an appropriate scalar field λ (we just need it to have a timelike gradient). Then the flow of U is approximated by the flow of U_n , and the latter being causally decreasing implies the same for the former. \square

We define *causally increasing* for a timelike vector field U to mean $(\mathcal{L}_U g)(N, N) \geq 0$, with $>$ for the strict version (equivalently: $-U$ is (strictly) causally decreasing); and for a transformation of V to mean its inverse is causally decreasing (same for strict version). We call a vector field or its flow (*strictly*) *causally monotonic* if it is either (strictly) causally decreasing or (strictly) causally increasing. We then obviously have

Corollary 2.2. *A timelike vector field U is causally monotonic iff its forward flow $\{R_t\}$ is causally monotonic; if the vector field is strictly causally monotonic, then so is its forward flow.* \square

We note that strict causal monotonicity of the flow does not, in general, imply the same for the vector field: Consider the manifold $M = \{(t, x) \in \mathbb{R}^2 \mid t > -1\}$ with metric $g = (dx)^2 - (t^3 + 1)(dt)^2$, and with the vector field $U = \partial/\partial t$. Then the forward flow of U carries causal vectors to timelike vectors (one need check only the null vectors), so this is strictly causally decreasing. However, $\mathcal{L}_U g = -3t^2(dt)^2$, which vanishes at $t = 0$, so U is only causally decreasing, and not strictly so.

It is the causal monotonicity of the flow of a vector field which yields the desired global property of the corresponding foliation (by the integral curves of the vector field). In light of Corollary 2.2, this local property of the flow is easily detected by the causal monotonicity of the vector field, an infinitesimal property. As was shown in Theorem 7 of [H2], the explicit connection is this:

Theorem 2.3. *If U is a causally monotonic and complete timelike vector field in a chronological spacetime V , then the foliation \mathcal{F} of integral curves of U is half-omniscient. (In particular, if U is future-directed and causally decreasing, then \mathcal{F} is past-omniscient.)*

Proof. (slightly simplifying the proof in [H2]) Let p be any point in V and $\gamma : \mathbb{R} \rightarrow V$ any integral curve of U ; with U future-directed and causally decreasing, we need to show γ enters the future of p . Let $\sigma : [0, 1] \rightarrow V$ be a curve from p to $\gamma(0)$, and define $\alpha : \mathbb{R} \times [0, 1] \rightarrow V$ by $\alpha(t, s) = t \cdot \sigma(s)$; let $T = \alpha_*(\partial/\partial t)$ (this is actually U) and $S = \alpha_*(\partial/\partial s)$ (so both T and S are flow-invariant). For some constant $m > 0$, $(mT + S)_{(0,s)}$ is future-timelike for all $s \in [0, 1]$. It then follows from Theorem 2.1 that for all $t > 0$, $(mT + S)_{(t,s)} = R_{t*}(mT + S)_{(0,s)}$ is timelike. Thus, the integral

curve τ of $mT + S$, $\tau(t) = \alpha(mt, t)$, is future-timelike for $t \geq 0$. In particular, $\tau(1) = \gamma(m)$ is to the future of $\tau(0) = p$. \square

Finally, we obtain a result which substantially generalizes Corollary 1 of [H1] (announced as Theorem 4 of [HL]):

Corollary 2.4. *Let V be a chronological spacetime with a causally monotonic and complete timelike vector field U , and let Q be the space of integral curves of U ; then Q is a manifold and $V \cong \mathbb{R} \times Q$.*

Proof. Immediate from Theorem 1.2, Proposition 1.3, and Theorem 2.3. \square

3. OMNISCIENCE, THE SHAPE OF SPACE, AND CONFORMASTATIONARY SPACETIMES

The main burden of this paper is to outline some common situations in which there is a well-defined “shape of space”. The key notion is to think of an edgeless spacelike hypersurface as exemplifying a possible shape of space; if all such hypersurfaces must be diffeomorphic to one another, then the use of the definite article is justified: There is only one topology possible for something representing all of space. But what should “edgeless” mean in this regard? A number of definitions are possible, but the easiest one to work with is that of a properly embedded hypersurface.

Recall that if $i : M \rightarrow V$ is injective, then i is proper if and only if for any sequence $\{x_n\}$ in M , if $\{i(x_n)\}$ is a convergent sequence in V , then $\{x_n\}$ converges in M . M is a spacelike hypersurface if, for g being the spacetime metric on V , the pulled-back metric i^*g is a Riemannian metric on M ; and M is acausal if for no two points x and y of M , is $i(x)$ in the causal past of $i(y)$.

Definition. By a *potential shape of space* for a spacetime V we will mean a properly embedded hypersurface $S \subset V$ such that any acausal, properly embedded, spacelike hypersurface (not necessarily connected) in V must be diffeomorphic to S . We will call a potential shape of space for V an *actual shape of space* if we know that there actually exists an acausal, properly embedded, and spacelike hypersurface in V .

(We will see in section 4 that this definition is, in a sense, stronger than it appears: If “properly embedded” for M is weakened to “edgelessly immersed” using either of a couple of suitable notions of “edgeless”, then the definition is, in actuality, unchanged, as any such edgeless immersion must actually be a proper embedding.)

Note that we are not insisting that a merely potential shape of space for V be itself embeddable as an acausal or even spacelike hypersurface—just as an edgeless hypersurface. Clearly, any two actual shapes of space for a given spacetime must be diffeomorphic to one another, though that is not evident for a potential shape of space. But it seems that the weaker notion is one that occurs naturally.

Here is an example of a chronological (but not strongly causal) spacetime with a potential, but not actual, shape of space:

Put slits in \mathbb{R}^2 by deleting the following line segments: for every even number $2n$, each closed vertical interval of length 1 along $x = 2n$ with lower end at $y =$ an odd number; and for every odd number $2n + 1$, each closed vertical interval of length 1 along $x = 2n + 1$ with lower end at $y =$ an even number. Call this slit plane, with the Euclidean metric, N . Consider the \mathbb{Z} -action on the spacetime $P = \mathbb{L}^1 \times N$

defined by $(t.x, y) \cdot n = (t + 2n, x + 2n, y)$; since no curve in N between (x, y) and $(x + 2, y)$ has length as small as 2, this action does not move any point to one in its past or future. Thus $V = P/\mathbb{Z}$ is (just barely) chronological; it is not strongly causal. The integral curves of $\partial/\partial t$ in P form a timelike foliation, preserved by the \mathbb{Z} action; V inherits this foliation. The leaf space in P is N ; in V it is $Q = N/\mathbb{Z}$. As V is static-complete, Corollary 3.2 (below) tells us that Q is a potential shape of space for V . But although Q can be properly embedded in V via $[x, y] \mapsto [x, y, x]$ (square brackets denoting equivalence class under the group action), there is no acausal proper embedding of Q into V : Q is not an actual shape of space for V , and so its status as a potential shape of space is vacuous.

Conjecture. Section 4 will illustrate some of the stronger results obtainable when one insists on strong causality, not merely chronology. Perhaps that is the key element for a strong shape of space? This seems to be a reasonable conjecture: A potential shape of space for a strongly causal spacetime must be an actual shape of space.

The most salutary virtue of an omniscient foliation is that it provides a potential shape of space (as announced in [HL] as Theorem 2, where it was called simply “shape of space”):

Theorem 3.1. *Let V be a chronological spacetime with an omniscient foliation \mathcal{F} of timelike curves. Then the leaf space Q is a potential shape of space for V .*

Proof. By Proposition 1.3 and Theorem 1.2, Q is Hausdorff. As argued above (before Theorem 1.2), this implies that the line-bundle $\pi : V \rightarrow Q$ has a cross-section $\sigma : Q \rightarrow V$ (in the smooth category), and σ must be a proper embedding: Since $\pi \circ \sigma = 1_Q$, σ is an injective immersion; if $\{\sigma(q_n)\}$ converges to some $p \in V$, then $\{\pi(\sigma(q_n))\} = \{q_n\}$ converges to $\pi(p)$. (But there is no guarantee that σ is spacelike or, if it is, that $\sigma(Q)$ is acausal.)

Consider any acausal, spacelike, proper embedding $i : M \rightarrow V$, with $\dim(M) = \dim(V) - 1$; we will show that $\pi \circ i : M \rightarrow Q$ is a diffeomorphism.

Since $i(M)$ is acausal, $\pi \circ i$ must be injective (else two points of M will be mapped by i to the same foliate). Let g be the metric on V ; since i^*g is Riemannian, $\pi \circ i$ is an immersion (π_* kills only timelike vectors). Then, by invariance of domain, $(\pi \circ i)(M)$ is an open subset of Q , and $\pi \circ i$ is a diffeomorphism onto this image; all we need show is that $\pi \circ i$ has a closed image.

Let $\{x_n\}$ be a sequence in M with $\{\pi(i(x_n))\}$ converging to some point $q \in Q$. Let γ_n be the foliate corresponding to $\pi(i(x_n))$ and γ the foliate corresponding to q , parametrized so that they begin in $\sigma(Q)$: $\gamma_n(0) = \sigma(\pi(i(x_n)))$ and $\gamma(0) = \sigma(q)$. We can define a continuous function $\tau : V \rightarrow \mathbb{R}$ by measuring how far each point of V is from $\sigma(Q)$, in terms of the real action: For any $p \in V$, $\tau(p) \cdot \sigma(\pi(p)) = p$. If we can show that the sequence $\{\tau(i(x_n))\}$ has a convergent subsequence, then we will be done: If $\{\tau(i(x_n))\}$ converges to t , then $\{i(x_{n_k})\} = \{\tau(i(x_{n_k})) \cdot \sigma(\pi(i(x_{n_k})))\}$ converges to $t \cdot \sigma(q)$; since i is proper, this means $\{x_{n_k}\}$ converges to some $x \in M$, with $i(x) = t \cdot \sigma(q)$. Then $\pi(i(x)) = \pi(t \cdot \sigma(q)) = q$, and $q \in \pi(i(M))$.

To show that $\{\tau(i(x_n))\}$ has a convergent subsequence, we will look at the points $\{i(x_n)\}$ for n sufficiently large: Consider that γ must enter the future of $i(x_1)$ at some point p^+ in V . The boundary of $I^+(i(x_1))$ is a three-dimensional topological manifold B^+ through p^+ , transverse to foliates of \mathcal{F} ; accordingly, we can find a relatively compact neighborhood U of q in Q such that the foliates γ' corresponding

to points $q' \in U$ each intersect B^+ within a relatively compact neighborhood of p^+ . We can do the same thing with p^- being the point where γ enters $I^-(i(x_1))$ and B^- the boundary of that past: the foliates sufficiently close to γ (we shrink U if necessary) intersect B^- at points within a relatively compact neighborhood of p^- . Then we have that portion of $\hat{U} = \pi^{-1}(U)$ which is in neither $I^+(i(x_1))$ nor $I^-(i(x_1))$ as a relatively compact set W between B^+ and B^- . For n sufficiently large, $\pi(i(x_n)) \in U$, so $i(x_n) \in \hat{U}$. Furthermore, since $i(M)$ is acausal, we cannot have $i(x_n) \gg i(x_1)$ or $i(x_n) \ll i(x_1)$: For these sufficiently high n , $i(x_n)$ must lie in W . Since W is relatively compact, the numbers $\{\tau(i(x_n))\}$ must have a convergent subsequence. \square

(This proof actually goes through with a slightly strengthened definition of potential shape of space: that any achronal (instead of merely acausal), spacelike, properly embedded hypersurface be diffeomorphic to the shape of space. Acausal is used in the definition in order to gain desired strength for the definition of actual shape of space, as used in the next section.)

As a class of examples, consider conformastationary spacetimes: This means there is a timelike vector field U so that, with g the metric, $\mathcal{L}_U(g) = \lambda g$ for some scalar function λ (where \mathcal{L} denotes Lie derivative). We will call the spacetime conformastationary-complete if the conformal-Killing field U is a complete vector field.

Corollary 3.2. *Any chronological conformastationary-complete spacetime has a potential shape of space.*

Proof. Let U be the conformal-Killing field and g the metric; then $\mathcal{L}_U(g)$ vanishes on any null vector, so U is both causally decreasing and causally increasing. Therefore, by Theorem 2.3, the foliation \mathcal{F} generated by U is omniscient. By Theorem 3.1, the leaf space of \mathcal{F} is a potential shape of space for the spacetime. \square

A somewhat less general result was given in [GH], Theorem 2: In a chronological stationary spacetime which is timelike or null geodesically complete, any achronally embedded spacelike hypersurface which is closed as a subspace of the spacetime must be diffeomorphic to the space of stationary observers (i.e., the leaf space of the foliation generated by the Killing field).

For a class of conformastationary-complete spacetimes (more general than the stationary-complete spacetimes of [GH]), consider the warped product example from section 1: $(\alpha, \omega) \times K$ with spacetime metric $g = -(dt)^2 + r(t)^2 h$ for h a Riemannian metric on K . The property of being conformastationary is conformally invariant, so we can instead consider the metric $\bar{g} = -(dt/r(t))^2 + h = -(d\tau)^2 + h$, where $d\tau/dt = 1/r(t)$. Then $U = \partial/\partial\tau$ is clearly a Killing field for \bar{g} , hence, a conformal-Killing field for g . U is complete if and only if $\int_c^\omega 1/r = \infty$ and $\int_\alpha^c 1/r = \infty$ for some c between α and ω .

4. TWO-SHEET OMNISCIENCE AND ACTUAL SHAPES OF SPACE

This section will consider situations in which we hope to derive global properties of an immersed spacelike hypersurface—such as its being an actual shape of space for the ambient spacetime—from as little information as possible, such as the action of the immersion on the fundamental group. The key ideas are that of an immersed hypersurface being “edgeless” (in a more general sense than being properly embedded) and of the spacetime having a “timelike-contractible disk” bounding any

null-homotopic curve. These ideas were introduced in [H2], but will be repeated here.

We will use this general notion of edgelessness as a weaker hypothesis for potential shapes of space. So long as the hypersurface is assumed to have an acausal image, it will follow (using a result in [H2]) that an analysis on the level of the fundamental group is sufficient to determine whether or not the immersion provides an actual shape of space.

The more difficult trick will be to draw such a conclusion *without* making a causality assumption on the image. To do that will require a slightly strengthened version of omniscience and that the ambient spacetime be strongly causal, not merely chronological. Strong results from [H2] (descending from a series of earlier results in [H3, H4]), making use of timelike contractible disks, and slightly modified here, will be used to achieve the desired goal.

Let M be a manifold of dimension 1 less than that of the spacetime V , and let $i : M \rightarrow V$ be an immersion. This is an approximation to a spacelike hypersurface so long as i induces a Riemannian metric on M : Let g be the spacetime metric on V ; then we call i a *spacelike immersion* so long as the pulled back metric i^*g is Riemannian.

Let $i : M \rightarrow V$ be a spacelike immersion. One way of being convinced that M is immersed in an edgeless sort of fashion is to note that i^*g is complete. But since we are interested in conformally invariant notions here, we will generalize this to *conformally completable*: That means that for some positive function $\Omega : V \rightarrow \mathbb{R}^+$, $i^*(\Omega g)$ is complete. Another notion of edgelessness would be to have $i : M \rightarrow V$ be proper; but that is stronger than is actually required, as much can be proved merely from looking at curves (this is one of the major themes in [H2]). Accordingly, it is useful to consider the property of being *curve-proper*: This means that for any curve $c : [0, 1] \rightarrow M$, if its image in V , $i \circ c$, has an endpoint at 1—i.e., $\lim_{t \rightarrow 1} i(c(t))$ exists in V —then c has (in M) an endpoint at 1 also. These notions are both strictly weaker than proper (let i map the line into an asymptotically “horizontal” spacelike helix in the Minkowski cylinder, $i : \mathbb{R} \rightarrow \mathbb{L}^1 \times \mathbb{S}^1$ with $i : x \mapsto (\tanh(x/2), [x])$, where $[x]$ is the projection of x into the circle; then i is a spacelike immersion of codimension 1 which is curve-proper and conformally completable, but not proper). Either of these ideas is an acceptable notion for edgelessness:

Definition. A spacelike immersion is called *edgeless* if it is curve-proper or conformally completable.

This is the notion of edgelessness referred to in the previous section, just after the definition of shape of space. Theorem 2 in [H2] says that for an edgeless spacelike immersion $i : M \rightarrow V$ of codimension 1, if the image $i(M)$ is achronal, then $i(M)$ is actually a properly embedded hypersurface, and $i : M \rightarrow i(M)$ is a covering projection; an example is $i : \mathbb{R} \rightarrow \mathbb{L}^1 \times \mathbb{S}^1$ with $i : x \mapsto (0, [x])$. Thus, if S is an actual shape of space for V , then not only is any properly embedded, acausal, spacelike hypersurface diffeomorphic to S , but also any merely immersed, acausal, spacelike hypersurface must be as well, so long as it is edgeless—though we must take care to speak of the image of the immersion being diffeomorphic to S , as the domain may be a cover of S . (This is not really a stronger statement of the meaning of shape of space, as any such immersed hypersurface—or, at any rate, its image—must actually be embedded.)

Proposition 4.1. *Let V^n be a chronological spacetime possessing an omniscient foliation \mathcal{F} by timelike curves. Let $i : M^{n-1} \rightarrow V$ be an edgeless spacelike immersion with acausal image $i(M)$. Then $i_* : \pi_1(M) \rightarrow \pi_1(V)$ is injective; and if $i_*(\pi_1(M)) = \pi_1(V)$, then $i : M \rightarrow V$ is an actual shape of space for V .*

Proof. By Theorem 2 of [H2], we know that $i(M)$ is a properly embedded spacelike hypersurface. We know from Theorem 3.1 here that $Q = V/\mathcal{F}$ is a potential shape of space for V , so $i(M)$ is diffeomorphic to Q ; in fact, $\pi : i(M) \cong Q$ (where $\pi : V \rightarrow Q$ is projection). Thus, $\pi_1(i(M)) \cong \pi_1(Q) \cong \pi_1(V)$ (the last because π is a projection with contractible fibre). Since $i : M \rightarrow i(M)$ is a covering map, $i_* : \pi_1(M) \rightarrow \pi_1(i(M))$ is injective, and it is surjective if and only if $i : M \rightarrow i(M)$ is a homeomorphism.

Since $\pi : i(M) \rightarrow Q$ and $\pi : V \rightarrow Q$ both induce isomorphisms of fundamental groups, we can translate the results above into statements about $i : M \rightarrow V$: $i_* : \pi_1(M) \rightarrow \pi_1(V)$ is injective, and it is also surjective if and only if $i : M \rightarrow i(M)$ is a homeomorphism. In the latter case, $\pi \circ i : M \rightarrow Q$ is also a homeomorphism, so M is a potential shape of space for V (since Q is). Furthermore, we then have M properly embedded in V via i in an acausal manner (since $i(M)$ is properly embedded and i is a homeomorphism onto its image), so M is an actual shape of space. \square

Edgelessly immersed spacelike hypersurfaces are looked at in detail in [H2] in the context of spacetimes which, broadly speaking, are not spacelike at timelike infinity. The key idea (in a nutshell) is that disks be suffered to exist long enough to shrink to a point. More precisely: We need that any null-homotopic loop in the spacetime be the boundary of a disk which is timelike-contractible:

Definition. A *timelike-contractible disk* in a spacetime V is an immersion $B : (-1, 1) \times D^2 \rightarrow V$, where D^2 is the closed disk in the plane, such that

- (1) for any $p \in D^2$, $B(-, p) : (-1, 1) \rightarrow V$ is a timelike curve,
- (2) B extends continuously to $[-1, 1] \times D^2$, and
- (3) $B(1, D^2)$ and $B(-1, D^2)$ are single points.

A timelike-contractible disk B *spans* a loop σ if σ is the boundary of $B(0, D^2)$. (Somewhat more precisely: This is a disk in $V - B(0, D^2)$ —together with a timelike contraction of the disk; the disk itself need not have any particular causal nature.)

As is shown in [H2], timelike-contractible disks are the key to having a spacetime sufficiently well behaved at timelike infinity: sufficiently well that edgeless spacelike immersions of codimension 1 are actually proper embeddings, so long as they do the right thing on the level of the fundamental group; specifically, one must have a timelike-contractible disk spanning each null-homotopic loop. But how can one tell if a spacetime has this property? Omniscient foliations of the sort considered here provide an easy answer—almost. We need to strengthen the notion of omniscience just a bit:

We need to specialize our consideration of a foliation \mathcal{F} to those foliates which intersect any particular curve in the spacetime V . This can be expressed in terms of timelike 2-surfaces in V : Given a non-degenerate (but possibly self-intersecting) curve $c : (a, b) \rightarrow V$, we consider P_c , roughly the union of all the foliates which contain any $c(s)$, parametrized by s . We almost can look at this as an immersed 2-surface by considering $i : \mathbb{R} \times (a, b) \rightarrow V$ defined by $i(t, s) = t \cdot c(s)$; but this

fails to be an immersion where $\dot{c}(s)$ is parallel to the foliate through $c(s)$. So instead consider the projection $\pi \circ c$; so long as c is not wholly lying along a single foliate, this curve can be parametrized as a non-degenerate curve $\delta : (\alpha, \beta) \rightarrow Q$. Then P_c is $\pi^{-1}(\text{Im}(\delta))$, expressed as an immersion via $i : \mathbb{R} \times (\alpha, \beta) \rightarrow V$ with $i(t, s) = t \cdot \sigma(\delta(s))$ (where $\sigma : Q \rightarrow V$ is a cross-section of π , as before); then P_c is a timelike 2-sheet in V (exceptionally: if c lies along a single foliate γ , then P_c is just γ). The foliation \mathcal{F} induces a foliation \mathcal{F}_c on P_c (in the exceptional case \mathcal{F}_c is just the single foliate γ).

Definition. A foliation \mathcal{F} of timelike curves in a spacetime V will be called *2-sheet omniscient* if for every non-degenerate curve c in V , not lying along a single foliate, the induced foliation \mathcal{F}_c is omniscient in the timelike 2-sheet P_c . (This is the same as saying that in the spacetime $\mathbb{R} \times (\alpha, \beta)$ with metric $i^*g - g$ the spacetime metric in V —the foliation $\{\gamma_s : t \mapsto (t, s) \mid \alpha < s < \beta\}$ is omniscient.)

This actually is a stronger notion of omniscience, as can be seen by considering the following spacetime: Start with \mathbb{L}^3 (metric $-dt^2 + dx^2 + dy^2$), but in the region $\{|y| < 1\}$, narrow the lightcones in the x -direction so as create “particle horizons” in the $y = \text{constant}$ planes, i.e., the null curves in those planes have vertical asymptotes. This can be done, for instance, with metric $-dt^2 + f(t, y)^2 dx^2 + dy^2$, where

$$f(t, y) = \begin{cases} (1 - y^2)t^2 + 1, & |y| < 1 \\ 1, & |y| \geq 1 \end{cases}.$$

Let \mathcal{F} be the foliation of t -curves, $\{x = x_0, y = y_0\}$. Then the induced foliation on each $\{y = y_0\}$ plane, with $|y_0| < 1$, is not omniscient (for instance, in $\{y = 0\}$, the null curves through $(0, 0, 0)$ have asymptotes at $x = \pm\pi/2$); but in the entire spacetime, \mathcal{F} is omniscient, since a timelike curve exists from any point (t_0, x_0, y_0) to any foliate $\{x = x_1, y = y_1\}$, even in the curved region $|y| < 1$, by first traveling in $x = x_0$ to the flat region, travelling there to $x = x_1$, then back along $x = x_1$ to the foliate.

It is interesting to note that this version of omniscience is inherited by covering spaces: If \tilde{V} is the universal covering space of the spacetime V , and V has a foliation \mathcal{F} by timelike curves, then \mathcal{F} induces a foliation $\tilde{\mathcal{F}}$ of timelike curves on \tilde{V} (lifting via $P : \tilde{V} \rightarrow V$, which is locally a diffeomorphism); and if \mathcal{F} is 2-sheet omniscient, then so is $\tilde{\mathcal{F}}$. This can be seen by considering the timelike 2-sheet $\tilde{\Pi}$ in \tilde{V} generated by a curve $\tilde{c} : [0, 1] \rightarrow \tilde{V}$ and trying to find a timelike curve from $\tilde{c}(0)$ to the foliate $\tilde{\gamma}_1$ ($\tilde{\gamma}_s$ being the foliate through $\tilde{c}(s)$). In $\Pi = P(\tilde{\Pi})$, there is a timelike curve δ from $c(0)$ to γ_1 ($c = P \circ \tilde{c}$, $\gamma_s = P \circ \tilde{\gamma}_s$); and δ can be expressed as $\delta(s) = \lambda(s) \cdot c(s)$ for some function $\lambda : [0, 1] \rightarrow \mathbb{R}$. The loop τ_1 formed by c , $\gamma_1|_{[0, \lambda(1)]}$, and δ is null-homotopic, as provided by the family of loops τ_s formed by $c|_{[0, s]}$, $\gamma_s|_{[0, \lambda(s)]}$, and $\delta|_{[0, s]}$. Thus, the loop τ_1 lifts to a loop $\tilde{\tau}_1$ in \tilde{V} , lying in $\tilde{\Pi}$, where its timelike portion δ goes from $\tilde{c}(0)$ to $\tilde{\gamma}_1$.

(Ordinary omniscience is not necessarily inherited by covering spaces: Consider as V the same spacetime as just above, restricted to $y > 0$, and with the line $\{x = 0, y = 1/2\}$ removed. In the universal covering space \tilde{V} , the induced foliation $\tilde{\mathcal{F}}$ is not omniscient, since going, in V , from $(0, -1, 1/4)$ around the missing line to get to the flat region, and back to the foliate $\{x = 1, y = 1/4\}$ is not in the

same homotopy class as staying in $\{y = 1/4\}$; thus, the lift in \tilde{V} reaches a different foliate.)

Proposition 4.2. *Let V be a spacetime with a foliation \mathcal{F} by timelike curves, which is 2-sheet omniscient. Then for every null-homotopic loop σ in V , there is a timelike-contractible disk spanning σ .*

Proof. We need work only in the differentiable category.

Let $\sigma : \mathbb{S}^1 \rightarrow V$ be a null-homotopic loop in V ; then there is a disk spanning σ , which we can take to be an immersion $b : D^2 \rightarrow V$, where D^2 is the closed unit disk in the plane, with the restriction of b to the boundary of D^2 being σ .

Let $\pi : V \rightarrow Q$ be the projection to the leaf space of \mathcal{F} . In virtue of \mathcal{F} being 2-sheet omniscient, we know the following: For any curve $c : [0, 1] \rightarrow Q$ and for any point $p \in \pi^{-1}(c(0))$, there is a unique future-null curve $c_p^+ : [0, 1] \rightarrow V$ which is a lift of c starting at p , i.e., $c_p^+(0) = p$ and $\pi \circ c_p^+ = c$, and also a unique past-null lift c_p^- of c starting at p . We know this because the induced foliation on the 2-sheet $P = \pi^{-1}(c)$ is omniscient, so the foliate $\gamma_1 = \pi^{-1}(c(1))$ enters both the past and future of p in terms of the Lorentz manifold P (more precisely: in terms of the Lorentz manifold $\mathbb{R} \times [0, 1]$, immersed with image P in V); at some point, it crosses the null curve in P from p . Furthermore, if we allow the curve c and the point p to vary, then $c_p^+(1)$ and $c_p^-(1)$ vary continuously with c and p in the C^1 topology, since c_p^+ and c_p^- are given as solutions to differential equations with these as input data.

Let us parametrize the disk D^2 by radial segments $\{r_\theta : [0, 1] \rightarrow D^2 \mid \theta \in \mathbb{S}^1\}$, with $r_\theta(0)$ on the boundary (at position θ) and $r_\theta(1)$ at the center. For each $\theta \in \mathbb{S}^1$, let $c_\theta = \pi \circ b \circ r_\theta$; then the future-null lifts of the images of r_θ , starting at the corresponding points of σ , vary continuously, i.e., the points $\{(c_\theta)_{\sigma(\theta)}^+(1)\}$ vary continuously in θ . Let γ be the foliate through $b(0)$; then there is a point p^+ on γ that is to the future of each $(c_\theta)_{\sigma(\theta)}^+(1)$. Similarly, there is a point p^- on γ to the past of each $(c_\theta)_{\sigma(\theta)}^-(1)$.

Consider each $P_\theta = \pi^{-1}(c_\theta)$; more precisely, P_θ is $\mathbb{R} \times [0, 1]$ with metric pulled back from the immersion $i_\theta : \mathbb{R} \times [0, 1] \rightarrow V$ defined via c_θ , i.e., $i_\theta(t, s) = t \cdot c_\theta(s)$. The points p^+ and p^- in V are represented by the same points $(t^+, 1)$ and $(t^-, 1)$ in each P_θ , while $(0, 0)$ in P_θ represents $\sigma(\theta)$ in V . There is a future-timelike curve δ_θ^0 in P_θ from $(t^-, 1)$ through $(0, 0)$ to $(t^+, 1)$; we can parametrize this as $\delta_\theta^0 : [-1, 1] \rightarrow P_\theta$ with $\delta_\theta^0(-1) = (t^-, 1)$, $\delta_\theta^0(0) = (0, 0)$, and $\delta_\theta^0(1) = (t^+, 1)$. Then we can fill in the portion of P_θ between δ_θ^0 and $\mathbb{R} \times \{1\}$ with a timelike foliation $\{\delta_\theta^s : [-1, 1] \rightarrow P_\theta \mid 0 \leq s \leq 1\}$ with $\delta_\theta^s(-1) = (t^-, 1)$, $\delta_\theta^s(0) = (0, s)$, and $\delta_\theta^s(1) = (t^+, 1)$, where δ_θ^1 runs along $\mathbb{R} \times \{1\}$.

(In case c_θ is degenerate—i.e., $b \circ r_\theta$ lies along a single foliate γ_θ —then P_θ is just \mathbb{R} , mapped into V as γ_θ . In that case, $(c_\theta)_{\sigma(\theta)}^+$ and $(c_\theta)_{\sigma(\theta)}^-$ are each the degenerate curve constant at $\sigma(\theta)$. The curves δ_θ^s are each a map taking -1 to t^- , 0 to 0 , and 1 to t^+ .)

We can form these various δ_θ^s in a manner that is differentiable in θ . Putting them all together then yields the map $B : [-1, 1] \times [0, 1] \times \mathbb{S}^1 \rightarrow V$ given by $B(t, s, \theta) = i_\theta(\delta_\theta^s(t))$, which is an immersion except at the various boundaries. (In case some c_θ is degenerate—which can happen only for isolated values of θ —we can vary the map B in a neighborhood $[-1, 1] \times [0, 1] \times \{\theta\}$ so as to preserve its status

as an immersion, while keeping $B(-, s, \theta)$ a timelike curve from p^- to p^+ .) Clearly, B is a timelike contraction of a disk spanning σ . \square

All we need do now to invoke the theorems of [H2] is upgrade the chronology assumption on V to strong causality.

Theorem 4.3. *Let V^n be a strongly causal spacetime possessing a 2-sheet omniscient foliation \mathcal{F} of timelike curves. Let $i : M^{n-1} \rightarrow V$ be an edgeless spacelike immersion. Then $i_* : \pi_1(M) \rightarrow \pi_1(V)$ is injective; and if $i_*(\pi_1(M)) = \pi_1(V)$, then $i : M \rightarrow V$ is an actual shape of space for V .*

Proof. By Proposition 4.2, every null-homotopic loop in V is spanned by a timelike-contractible disk. Corollary 5 of [H2] says precisely that for a strongly causal spacetime V with that property, for any codimension-1 edgeless spacelike immersion $i : M \rightarrow V$, $i_* : \pi_1(M) \rightarrow \pi_1(V)$ is necessarily injective; and that if i_* is surjective, then i must be an achronal proper embedding.

In fact, M is diffeomorphic to the leaf space Q of the foliation \mathcal{F} : Theorem 3.1 tells us Q is a potential shape of space for V and, in fact, that $\pi \circ i : M \rightarrow Q$ is a diffeomorphism (the parenthetical comment directly following Theorem 3.1 indicates that we need have M only achronally embedded, not acausally).

Suppose N is any acausal and properly embedded hypersurface in V ; then another application of Theorem 3.1 shows N is diffeomorphic (via π) to Q , hence, to M . This shows M is a potential shape of space for V (but since we have not yet shown M to be acausally embedded, we cannot yet say it is an actual shape of space).

All that remains is to show that M is, indeed, acausally embedded in V , not just achronally. This requires a slight strengthening of the results in [H2] and its predecessors [H3, H4]. This is essentially technical in nature, involving one small piece of substantive work and a deal of minor bookkeeping.

Specifically, what is needed is to strengthen Corollary 5 of [H2], which says that if V^n is strongly causal and has every null-homotopic loop spanned by a timelike-contractible disk, then any edgeless spacelike immersion $i : M^{n-1} \rightarrow V$ must be injective on the level of the fundamental group, and if $i_* : \pi_1(M) \rightarrow \pi_1(V)$ is onto, then i must be an achronal proper embedding. This must be strengthened to say that that i must be acausal, not merely achronal. Backtracking through [H4] and [H3], one sees that the crucial step comes in Theorem 3 of [H3].

An argument is given there that says that in a Lorentz manifold homeomorphic to the plane, a timelike and a spacelike curve cannot intersect twice (since from any given point p , the null cone from p separates the timelike curves issuing from p from the spacelike curves issuing from p). But this argument applies equally well to a causal and a spacelike curve. That observation then propagates through Proposition 4 and Theorems 5 and 6 of [H3], Proposition 1, Theorem 2, and Corollary 3 of [H4], and Theorem 3, Corollary 4, and Corollary 5 of [H2] (with appropriate changes being made in a handful of definitions along the way).

The strengthened Corollary 5 of [H2] then does the trick here: From Proposition 4.2, we know every null-homotopic loop in V is spanned by a timelike-contractible disk. Thus, i embeds M into V in an acausal manner. \square

As an application of the ideas from this and the previous section, consider strongly causal, conformastationary-complete spacetimes: We know from Corollary 3.2 that the conformastationary-observer space is a potential shape of space

for such a spacetime; but with strong causality, more information is available to us (this was announced as Theorem 3 in [HL]):

Theorem 4.4. *Let V^n be a strongly causal, conformastationary-complete spacetime, and let Q be the conformastationary-observer space, i.e., the leaf space of the foliation defined by the conformal-Killing vector field, with $\pi : V \rightarrow Q$ the projection. Let $i : M \rightarrow V$ be an edgelessly immersed spacelike hypersurface. Then $\pi \circ i : M \rightarrow Q$ is a covering projection with fibre $\pi_1(V)/i_*(\pi_1(M))$; in particular, if i_* is onto, then M is homeomorphic to Q , and M is acausally embedded.*

Proof. As indicated in the proof of Corollary 3.2, we know that the foliation \mathcal{F} defined by the conformal-Killing field is omniscient. But to apply Theorem 4.3, we need to show it is 2-sheet omniscient. So let c be any non-degenerate curve in V (not consisting of only a single conformastationary-observer orbit), and let P_c be the immersed timelike 2-surface defined by the orbits passing through c (essentially, $P_c = \pi^{-1}(\pi(c))$), with \mathcal{F}_c the induced foliation from \mathcal{F} and g_c the induced metric (in detail: $\pi \circ c$ is given by a non-degenerate curve $\delta : (\alpha, \beta) \rightarrow Q$, $P_c = \mathbb{R} \times (\alpha, \beta)$, we have an immersion $j_c : P_c \rightarrow V$ via $j(t, s) = t \cdot \sigma(\delta(s))$ for some chosen cross-section $\sigma : Q \rightarrow V$ of π , \mathcal{F}_c is the family $\{\mathbb{R} \times \{s\} \mid \alpha < s < \beta\}$, and $g_c = (j_c)^*(g)$, where g is the spacetime metric on V). Then (P_c, g_c) is a conformastationary-complete spacetime in its own right; thus, as in Corollary 3.2, \mathcal{F}_c is omniscient in P_c .

Thus, Theorem 4.3 applies to V : We know $i_* : \pi_1(M) \rightarrow \pi_1(V)$ is injective, and that if i_* is onto, then i is an acausal embedding and, by Theorem 3.1, $\pi \circ i : M \rightarrow Q$ is a homeomorphism.

But what happens if i_* is not onto?

Let \tilde{V} be the universal covering space of V , with $p_V : \tilde{V} \rightarrow V$ the projection; then \tilde{V} is also conformastationary-complete. With \tilde{M} the universal covering space of M (with projection $p_M : \tilde{M} \rightarrow M$), we have an induced immersion $\tilde{i} : \tilde{M} \rightarrow \tilde{V}$ ($p_V \circ \tilde{i} = i \circ p_M$). Clearly, \tilde{i} inherits being edgeless and spacelike; thus, by Theorem 4.3, it is an acausal embedding.

We have the following maps:

The map $i : M \rightarrow V$ is covered by the map $\tilde{i} : \tilde{M} \rightarrow \tilde{V}$, in the sense that $p_M : \tilde{M} \rightarrow M$ and $p_V : \tilde{V} \rightarrow V$ are both principal covering maps (i.e., the fibre is a group and projection is by identification under the group action) with structure groups, respectively, $\pi_1(M)$ and $\pi_1(V)$, and that the group actions are preserved through the action of $i_* : \pi_1(M) \rightarrow \pi_1(V)$, i.e., for $g \in \pi_1(M)$ and $\tilde{x} \in \tilde{M}$, $i(p_M(g \cdot \tilde{x})) = p_V((i_*g) \cdot \tilde{i}(\tilde{x}))$.

The projection to observer-space $\pi : V \rightarrow Q$ has its mirror, projection to observer-space $\tilde{\pi} : \tilde{V} \rightarrow \tilde{Q}$. The principal covering map $p_V : \tilde{V} \rightarrow V$ is easily seen to induce a map $p_Q : \tilde{Q} \rightarrow Q$, which is also a principal covering map; since \tilde{Q} is simply connected, p_Q must actually be the universal covering map for Q with structure group $\pi_1(Q)$. Thus, $\tilde{\pi}$ covers π in the sense that for $g \in \pi_1(V)$ and $\tilde{x} \in \tilde{V}$, $\pi(p_V(g \cdot \tilde{x})) = p_Q(\pi_*(g) \cdot \tilde{\pi}(\tilde{x}))$. (Both π and $\tilde{\pi}$ are principal fibre bundles with structure group \mathbb{R} , and $p_Q(\tilde{\pi}(t \cdot \tilde{x})) = \pi(t \cdot p_V(\tilde{x}))$.)

From Theorem 4.3 applied to \tilde{i} , we know $\tilde{\pi} \circ \tilde{i} : \tilde{M} \rightarrow \tilde{Q}$ is a homeomorphism. Thus, the map $p = p_Q \circ (\tilde{\pi} \circ \tilde{i}) : \tilde{M} \rightarrow Q$ is a universal covering map for Q ; since \tilde{i} covers i and $\tilde{\pi}$ covers π , we know $\tilde{\pi} \circ \tilde{i}$ covers $\pi \circ i$, i.e., p factors as $(\pi \circ i) \circ p_M$. The action of the map p_M is to factor out the action of $\pi_1(M)$ from p as a universal covering map ($\pi_1(M)$ can be regarded as a subgroup of $\pi_1(Q)$ via $\pi_* \circ i_*$, since π_* is

injective due to general covering map properties, and i_* is injective due to Theorem 4.3). That leaves $\pi \circ i$ as the remaining portion of the principal covering map p , so $\pi \circ i$ is also a covering map with fibre $\pi_1(Q)/\pi_1(M)$, i.e., $\pi_1(V)/i_*(\pi_1(M))$. \square

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